The General Solution of Coupled Riccati Equations Based on Nonlinear Superposition

Jinfeng Ding[1], Mingliang Zheng[2*]

[1]*School of Intelligent Equipment Engineering, Taihu University of Wuxi, Wuxi 214063, China*

[2]*School of Mechanical and Electrical Engineering, Huainan Normal University, Huainan 232038, China*

*E-mail: liangmingzheng@hnnu.edu.cn

Abstract: Due to the fact that the Riccati equations are nonlinear equations, it is difficult to obtain its analytical solution by using commonly used elementary integration methods. We discuss the calculation of general solution for a class of coupled Riccati equations. Firstly, the nonlinear superposition theorem of ordinary differential equations is given; Then the general solution expression and properties of coupled Riccati equations are obtained on the basis of known particular solutions; Finally, a special type of Riccati equations are calculated by the nonlinear superposition formula, which shows that the nonlinear superposition is effective for the integration of nonlinear equations. Nonlinear superposition method has universal significance in constructing new composite exact solutions for nonlinear evolution equations.

Keywords: coupled Riccati equations; nonlinear superposition; general solution; composite exact solutions

INTRODUCTION

Riccati equations [1] are a class of important differential equations, which is widely used in mathematics and engineering science. For example, it can be used to prove that the solution of Bessel equation is not an elementary function [2]; It can give the type of infinite sequence exact solution for KdV equation [3]; Many models in modern control theory and vector field branching theory are in the form of Riccati equations [4]. As everyone knows, the nonlinear Riccati differential equation $\frac{dx}{dt}=T_1(t)+T_2(t)x+T_3(t)x^2, T_i(t)\neq 0 \in C(I)$ has been proved to be unable to be expressed analytically by elementary function [5], however, it still has many excellent properties, which is very valuable for solving some special Riccati equations.

The Riccati equations are nonlinear equations with high order and coupled variables in matrix form, making it very difficult to solve. In general, it is impossible to find analytical expressions directly represented by coefficient matrices and weighting matrices, and currently, most people use numerical methods [6-7] to solve problems through computers. In the past fifty years, more than ten different methods for solving the Riccati equation have been proposed, such as direct integration, complex exponential method, iterative method [8-10]. However, these methods generally have drawbacks such as large computational complexity and poor convergence, making it difficult to obtain satisfactory results.

The principle of superposition of differential equations is a very important principle in differential equation theory, which can help us solve some nonlinear differential equations. The core idea of the superposition principle is to split a differential equation into several simpler differential equations, and then linearly stack the solutions of these simple differential equations to obtain the solution of the entire differential equation. Lie [11] found the nonlinear superposition formula in the integration theory of linear partial differential equation. In recent years, the nonlinear superposition principle of differential equation has attracted more and more attention. In reference [12], it studied the Darboux transformations, the Backlund transformations and the superposition formula for three coupled systems; In reference [13], it studied the backlund transformation and the nonlinear superposition formula of of nonlinear equation. In reference [14], it studied the backlund transformation, nonlinear superposition formula and infinite conservation laws of Benjamin equation. On the one hand, these studies greatly promote the integration theory of nonlinear differential equations, on the other hand, the nonlinear superposition theory is further developed. In order to obtain the general solution form of a class of coupled Riccati equations, the nonlinear superposition formula of the general solution is given by using the structural characteristics and special solutions of the system. Its key technology is the invariants of Lie algebraic, and a specific example is given for illustration and verification.

1. THE NONLINEAR SUPERPOSITION THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

Please refer to reference [15], if the ordinary differential equations:

$$\frac{dx^i}{dt} = f^i(t.\mathbf{x})(i=1,2...,n) \tag{1}$$

Which have a basement of solutions, it must have the form of generalized separated variables:

$$\frac{dx^i}{dt} = T_1(t)\xi_1^i(\mathbf{x}) + ... + T_r(t)\xi_r^i(\mathbf{x}), (i=1,2...,n) \tag{2}$$

And the operators $X_\alpha = \xi_\alpha^i \frac{\partial}{\partial x^i}, \alpha = 1,..,r$ must constitute a finite dimensional- r Lie Algebra L_r (also known as vessiot-Guldberg Lie algebra). Here the coefficient $T_r(t)$ is a arbitrary function of t .

Nonlinear superposition theorem [15]: if the general solution of equations (2) is $\mathbf{x} = (x^1,..., x^n)$, and it has special solutions with m -number $\mathbf{x}_1 = (x_1^1,..., x_1^n),..., \mathbf{x}_m = (x_m^1,..., x_m^n)$ and arbitrary constants with n -number $C_1,\cdots,C_n$, which are satisfied:

$$J_i(x^1,...,x^n, x_1^1,...,x_1^n,...,x_m^1,...,x_m^n) = C_i \tag{3}$$

Here J_i is the group invariants of $X_1,..., X_r$ constructed by $x^i, x_1^i,..., x_m^i$, $\det(\partial J_i / \partial x^k) \neq 0$, so:

$$\xi_\alpha^i(\mathbf{x})\frac{\partial J}{\partial x^i} + \xi_\alpha^i(\mathbf{x}_1)\frac{\partial J}{\partial x_1^i} + \cdots + \xi_\alpha^i(\mathbf{x}_m)\frac{\partial J}{\partial x_m^i} = 0, (\alpha = 1,...,r) \tag{4}$$

Then the nonlinear superposition formula of the general solution is:

$$x^i = \varphi^i(\mathbf{x}_1,...,\mathbf{x}_m; C_1,...,C_n)(i=1,...,n) \tag{5}$$

There is restriction $nm \geq r$, and the special solutions $\mathbf{x}_1,...,\mathbf{x}_m$ is a basement of solutions.

2. THE NONLINEAR SUPERPOSITION OF UNCOUPLED RICCATI EQUATION

The uncoupled Riccati equation with constant coefficients is as follows:

$$\frac{dx}{dt} = a(t) + b(t)x + c(t)x^2 \tag{6}$$

Nonlinear superposition theorem: when $x_{k-1}(t)$ and $x_{k-2}(t)$ are the solutions of Equation (6), the following $x_k(t)(k=2,...)$ are also the solutions of Equation (6):

$$x_k(t) = \frac{N_1[\mp 2a^2A + [\mp 2a^2C + N_3 x_{k-1}(t)]x_{k-2}(t)]}{N_2 \pm N_4[Abx_{k-2}(t) + [Ab - aC + Acx_{k-2}(t)]x_{k-1}(t)]} \tag{7}$$

Here,

$$\begin{aligned}
N_1 &= (b \pm \sqrt{b^2 - 4ac})^2, \\
N_2 &= 4a[\pm 2Ab^3 \mp 5aAbc \mp ab^2C \pm 2a^2cC - \sqrt{b^2-4ac}(-2Ab^2 + aAc + abC)], \\
N_3 &= [\pm Ab^2 \mp 2aAc \mp abC + \sqrt{b^2-4ac}(Ab - aC)], \\
N_4 &= [\pm Ab^2 \mp 2aAc \mp abC + \sqrt{b^2-4ac}(Ab - aC)].
\end{aligned}$$

and A, B, C are arbitrary constants.

Further, when $x_{k-1}(t)$, $x_{k-2}(t)$ and $x_{k-3}(t)$ are the solutions of Equation (6), the following $x_k(t)(k=3,\ldots)$ are also the solutions of Equation (6):

$$x_k(t)=\frac{aAx_{k-3}(t)-[P_2(t)+(aB+bC)x_{k-3}(t)]x_{k-1}(t)+P_1(t)x_{k-2}(t)}{aDx_{k-3}(t)+[P_3(t)+cCx_{k-3}(t)]x_{k-1}(t)+[aB-c(A+C)x_{k-3}(t)]x_{k-2}(t)} \tag{8}$$

Here, $\begin{array}{l} P_1(t)=aC+[a(B+D)+b(C+A)]x_{k-3}(t), \\ P_2(t)=a(C+A)+(aD+bA)x_{k-2}(t), \\ P_3(t)=-a(B+D)+cAx_{k-2}(t). \end{array}$, and A,B,C,D are arbitrary constants.

Theorem 1: when $x_1(t),x_2(t),x_3(t)$ are the three divergent solutions of Equation (6), so the general solution of the Riccati equation (6) is:

$$x(t)=\frac{(x_3-x_1)(x_1-x_2)}{C(x_3-x_2)-(x_3-x_1)}+x_1 \tag{9}$$

Prove: Easy to know: $\dot{x}_1=a+bx_1+cx_1^2,\dot{x}_2=a+bx_2+cx_2^2,\dot{x}_3=a+bx_3+cx_3^2$.

So, $2cx_3+b=c(x_3-x_2)+\dfrac{\dot{x}_3-\dot{x}_2}{x_3-x_2},2cx_3+b=c(x_3-x_1)+\dfrac{\dot{x}_3-\dot{x}_1}{x_3-x_1}$.

So, $c(x_1-x_2)=\dfrac{\dot{x}_3-\dot{x}_1}{x_3-x_1}-\dfrac{\dot{x}_3-\dot{x}_2}{x_3-x_2}$.

So, the general solution is:

$$x=\frac{(x_1-x_2)e^{\int c(x_1-x_2)dt}}{C-e^{\int c(x_1-x_2)dt}}+x_1=\frac{(x_1-x_2)e^{\int(\frac{\dot{x}_3-\dot{x}_1}{x_3-x_1}-\frac{\dot{x}_3-\dot{x}_2}{x_3-x_2})dt}}{C-e^{\int(\frac{\dot{x}_3-\dot{x}_1}{x_3-x_1}-\frac{\dot{x}_3-\dot{x}_2}{x_3-x_2})dt}}+x_1=\frac{(x_1-x_2)e^{\ln(x_3-x_1)-\ln(x_3-x_2)}}{C-e^{\ln(x_3-x_1)-\ln(x_3-x_2)}}+x_1$$

$$=\frac{(x_3-x_1)(x_1-x_2)}{C(x_3-x_2)-(x_3-x_1)}+x_1$$

It can be seen that when the three specific solutions of Equation (6) are known, the general solution can be obtained without the need for integration.

3. THE NONLINEAR SUPERPOSITION OF COUPLED RICCATI EQUATIONS

The coupled Riccati equations are as follows:

$$\begin{cases} \dfrac{dx}{dt}=T_1(t)+2T_2(t)x+T_3(T)x^2, \\ \dfrac{dy}{dt}=T_2(t)y+T_3(t)xy. \end{cases} \tag{10}$$

The equations(10) are corresponding to $r=3,\xi_1^1(\mathbf{x})=1,\xi_2^1(\mathbf{x})=2x,\xi_3^1(\mathbf{x})=x^2,\xi_1^2(\mathbf{x})=0,\ \xi_2^2(\mathbf{x})=y,\ \xi_3^2(\mathbf{x})=xy$, and the operators are:

$$X_1=\frac{\partial}{\partial x},X_2=2x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},X_3=x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y} \tag{11}$$

It's easy to verify X_1, X_2, X_3 are linearly independent, and $\left[X_i, X_j\right] = c_{ij}^k X_k$, so X_1, X_2, X_3 generate a Lie algebra L_3.

According to the theorem 1, $2m \geq 3$, so $m = 2$, the common invariants of the operators are:

$$\frac{\partial J}{\partial x} + \frac{\partial J}{\partial x_1} + \frac{\partial J}{\partial x_2} = 0,$$
$$2x\frac{\partial J}{\partial x} + y\frac{\partial J}{\partial y} + 2x_1\frac{\partial J}{\partial x_1} + y_1\frac{\partial J}{\partial y_1} + 2x_2\frac{\partial J}{\partial x_2} + y_2\frac{\partial J}{\partial y_2} = 0, \tag{12}$$
$$x^2\frac{\partial J}{\partial x} + xy\frac{\partial J}{\partial y} + x_1^2\frac{\partial J}{\partial x_1} + x_1 y_1\frac{\partial J}{\partial y_1} + x_2^2\frac{\partial J}{\partial x_2} + x_2 y_2\frac{\partial J}{\partial y_2} = 0.$$

The equations (12) has a solution:

$$J_1 = \frac{yy_1}{x_1 - x} = C_1, J_2 = \frac{yy_2}{x_2 - x} = C_2 \tag{13}$$

Therefore, the expression of the general solution with using the two special solutions (x_1, y_1) and (x_2, y_2) is:

$$x = \frac{C_1 x_1 y_2 - C_2 x_2 y_1}{C_1 y_2 - C_2 y_1}, y = \frac{C_1 C_2 (x_2 - x_1)}{C_1 y_2 - C_2 y_1} \tag{14}$$

The equation (14) shows the general solution of the coupled Riccati equations is a nonlinear superposition of two special solutions. We only need two special solutions to get the general solution, which is different from the three special solutions (basement) required for the nonlinear superposition of the one-dimensional Riccati equation. In fact, the nonlinear superposition of one-dimensional Riccati equation $\frac{dx}{dt} = P(t) + Q(t)x + R(t)x^2$ is that the cross ratio of solutions does not depend on x ,that is $\frac{x - x_2(t)}{x - x_1(t)} : \frac{x_3(t) - x_2(t)}{x_3(t) - x_1(t)} = C$. So, we can infer that an important characteristic of coupled differential equations is to limit the dimension of the basement of solutions .

4. EXAMPLE ILLUSTRATION

Now we calculate the coupled and uncoupled Riccati equations.

(1) Setting $T_1(t) = \frac{2}{t^2}, T_2(t) = 0, T_3(t) = -1$, so there are two special solutions for coupled equations:

$$x_1 = \frac{2}{t}, y_1 = \frac{1}{t^2}; \quad x_2 = -\frac{1}{t}, y_2 = t \tag{15}$$

According to the equations (14), so the general solution is:

$$x = \frac{2C_1 + C_2 / t^3}{C_1 t - C_2 / t^2} = \frac{2t^3 + \frac{C_2}{C_1}}{t^4 - \frac{C_2}{C_1}t}, y = -\frac{3C_1 C_2 t}{C_1 t^3 - C_2} \tag{16}$$

This is a very excellent result. In reference [16], the general solution form of the first equation of in equations (10) is $x = \frac{2t^3 + C}{t(t^3 - C)}$ by Lie symmetry method, which is consistent with equations (16), so our nonlinear superposition formula is correct.

(2) Setting $T_1 = 0$, $T_3(t) = 0$, so the uncoupled Riccati equations (10) transform to linear equations, and its structural characteristics are:

$$r = 1, \xi_1^1(\mathbf{x}) = 2x, \xi_1^2(\mathbf{x}) = y.$$
$$X_1 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \quad (17)$$

So $2m \geq 1$, still taking $m = 2$, and the invariant equation of operator is:

$$2x\frac{\partial J}{\partial x} + y\frac{\partial J}{\partial y} + 2x_1\frac{\partial J}{\partial x_1} + y_1\frac{\partial J}{\partial y_1} + 2x_2\frac{\partial J}{\partial x_2} + y_2\frac{\partial J}{\partial y_2} = 0. \quad (18)$$

There is a set of solutions for equations (18):

$$J_1 = \frac{xy_2 - x_2 y}{x_1 y_2 - x_2 y_1} = C_1, J_2 = \frac{x_1 y - x y_1}{x_1 y_2 - x_2 y_1} = C_2 \quad (19)$$

As long as the special solutions (x_1, y_1) and (x_2, y_2) are not linearly independent, the general solution is:

$$x = C_1 x_1 + C_2 x_2, y = C_1 y_1 + C_2 y_2 \quad (20)$$

Obviously, the result is correct, because one of the important properties of homogeneous equations is to satisfy the linear superposition, and the linearly independent special solutions provide a basement of solutions.

To sum up, no matter the coupled nonlinear Riccati equations or the uncoupled linear Riccati equations, the expression of the general solution based on the nonlinear superposition principle is correct. Compared with other integral methods of the general solution, the nonlinear superposition is simple in calculation and formula, the general solution can be constructed only by using the special solutions.

CONCLUSION

This article mainly uses the Lie symmetric transformation to solve the coupled integrable Riccati equations. Based on the obtained Lie algebraic structure and invariant solutions, we construct the corresponding nonlinear superposition formula, and use the known special solutions to obtain the general solution of the original equations. We introduced the specific steps and criterion for solving the general solution of ordinary differential equations by nonlinear superposition principle. Research has found that as long as Riccati equations satisfy certain conditions (generalized separability and Lie algebra structure), and have some suitable special solutions, we can get its general solution by looking for invariants of Lie symmetric transformation operator. Meanwhile, we give the nonlinear superposition formula of general solution of a kind of coupled Riccati equations, which is similar to the nonlinear superposition property of Riccati equation with one-dimensional, but it reduces the number of special solutions needed, which is a major feature of coupled equations. The nonlinear superposition algorithm can be extended to other nonlinear systems.

REFRENCES

[1] Wang Gaoxiong. The Ordinary differential equation[M]. Beijing: Higher Education Press, 2004.(in Chinese)

[2] Guang Yunqing. The general solution of generalized Riccati equation[J]. Journal of Kaifeng Institute of Education, 2018,38(4):104-105.(in Chinese)

[3] Tao Ge Tusang. Nonlinear Superposition Formula of the SolutionsMto Riccati Equation and Its Application[J]. Journal of Inner Mongolia Normal University (Natural Science Edition), 2011, 40(3):207-212.(in Chinese)

[4] TobiasBreiten, KarlKunisch, LaurentPfeiffer. A reduction method for Riccati-based control of the Fokker-Planck equation[J]. IFAC-PapersOnLine, 2017,50(1):1631-1636.

[5] Ping Genjian. Integrating Factor Method of Seeking General Solution to One Kind of Riccati Differential Equation[J]. Journal of Xi' an University of Arts&Science(Nat Sci Ed), 2012,15(2):32-34.(in Chinese)

[6] ALHARBI A R , ALMATRIFI M B. New exact and numerical solutions with their stability for Ito integro-differential equation via Riccati-Bernoulli sub-ODE method[J]. Journal of Taibah University for Science, 2020(14):1447-1456.

[7] EI-TAWIL M A, BAHNASAWI A A, ABDEL-NABY A. Solving Riccati differential equation using Adwoman's decomposition method[J]. Appl Math Comput, 2004,(157): 503-514.
[8] KHALID H, JAWAD K K. An Analytic Solution for Riccati Matrix Delay Differential Equation using Coupled Homotopy-Adomian Approach[J]. Baghdad Science Journal, 2022,19(4): 800-804.
[9] GENG F. A modified variational iteration method for solving Riccati differential equations[J]. Comp. Math. with Application, 2010,60(7):1868-1872.
[10] DOHA E H, ABDELKAWY M A. Approximate solutions for solving nonlinear variable-order fractional Riccati differential equations[J]. Nonlinear Analysis:Modelling and Control, 2019,2(24):176-188.
[11] Sophus Lie. Allgemeine Untersuchungen über Differentialgleichungen, die eine continuirliche, endliche Gruppe gestatten[J]. Mathematische Annalen, 1885, 25 (1):71-151.
[12] Niu Xiaoxing. DarbouxTransformations, Backlund Transformations and Superposition Formula for Three Coupled Systems[D]. Beijing: China Mining University,2017.(in Chinese)
[13] Shan Hongli. Bäcklund transformation and superposition formula for solutions of nonlinear equations[D]. Inner Mongolia:Inner Mongolia Normal University, 2012.(in Chinese)
[14] Zhang Hongqing, Zhang Yufeng. BäcklundTransformation, Nonlinear Superposition Formulae and Infinite Conserved Laws of Benjamin Equation[J]. Applied Mathematics and Mechanics, 2001,22(10):1018-1022.(in Chinese)
[15] N. H. Ibragimov, M. Torrisi, A.Valenti. Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics[M]. Berlin:Springer, 2012.
[16] Nail H. Ibragimov. Differential Equations and Mathematical Physics Problems [M]. Beijing: Higher Education Press, 2013.(in Chinese)